TABLE III
FILTER SPECIFICATIONS ACHIEVED

	Filter Using 4-th Order Subfilters		Filter Using 6-th Order Subfilters	
Ripple in the Passband	0.88 dB		0.78 dB	
Minimum Stopband Attenuation	22.28 dB		23.29 dB	
Maximum Approximation Error in the Passband	Main Section:	0.2600	Main Section:	0.2143
	With One Correction Section:	0.0962	With One Correction Section:	0.1478
	With Two Correction Section:	0.1602	With Two Corriction Section:	0.0863
Maximum Approximation Error in the Stopband	Main Section:	0.0800	Main Section:	0.0762
	With One Correction Section:	0.0769	With One Correction Section:	0.1345
	With Two Correction Section:	0.0703	With Two Correction Section:	0.0685

On Factorization of a Subclass of 2-D Digital FIR Lossless Matrices for 2-D QMF Bank Applications

V. C. LIU AND P. P. VAIDYANATHAN

***Abstract*—The role of one-dimensional (1-D) digital FIR lossless matrices in the design of FIR perfect reconstruction QMF banks has been explored in several recent articles. Structures which can realize the complete family of FIR lossless transfer matrices have also been developed, with QMF application in mind. For the case of 2-D QMF banks, the same concept of lossless polyphase matrix has been used to obtain perfect reconstruction. However, the problem of finding a structure to cover all 2-D FIR lossless matrices of a given degree has not been solved. In this letter we make some progress in this direction. We obtain a structure which completely covers a well-defined subclass of 2-D digital FIR lossless matrices.**

The design of maximally decimated digital filter banks with perfect reconstruction has been addressed by a number of authors in recent years [1]–[3] along with 2-D extensions [4]–[6]. This letter is related to the method reported in [3] and the 2-D version reported in [6]. The notations we shall use are the same as those in [3] and [6]. Consider the M-channel maximally decimated QMF bank used in [3]. Let each analysis filter $H_k(z)$ be represented in its polyphase form as in [3] so that we can define the $M \times M$ polyphase matrix $E(z) = [E_{kn}(z)]$, which completely characterizes the analysis filters. The perfect reconstruction property in [3] was based on the losslessness property of $E(z)$. As a remainder, the FIR matrix $E(z)$ is said to be lossless if it satisfies the paraunitary property $\tilde{E}(z)E(z) = I$ where $\tilde{E}(z) \triangleq E_*^T(z^{-1})$ (subscript $*$ stands for coefficient conjugation only). On the unit circle, this property reduces to the unitariness of $E(e^{j\omega})$.

In [7], [8] structures for FIR lossless transfer matrices were developed based on the *discrete-time lossless lemma* (DTL Lemma [9]) which is a result about the state-space manifestation of losslessness. This lemma [9] merely says that $E(z)$ is lossless if and only if there exists a structure with state-space description A, B, C, D (notations as in [8]) such that the system matrix

$$R \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

is unitary. Because of the FIR nature of $E(z)$, this lemma leads to a structure which was exploited in [8] to optimize attenuation characteristics of the analysis filters $H_k(z)$. When the parameters of the structure are being optimized, the perfect reconstruction property remains intact because the losslessness of $E(z)$ is *structurally* guaranteed.

2-D QMF banks [4]–[6] find applications in subband coding of images. The results in [3] on perfect reconstruction has been extended to the 2-D case in [6] (for rectangular decimation) and in [10] (for general decimation matrices). The use of 2-D FIR lossless polyphase matrices in obtaining perfect reconstruction has been recognized in [6], [10]. However, unlike in the 1-D case, it has not been possible in the 2-D case to obtain a complete structural characterization of $M \times M$ FIR lossless transfer matrices. The main difficulty is because there is no 2-D equivalent of the *lossless lemma*. In other words, given a 2-D lossless system, we are not guaranteed to find a structure whose system matrix is unitary.

For the 1-D case, a second alternative form of structures were introduced in [11] which does not require a state-space formulation. This form also completely covers all $M \times M$ FIR causal lossless systems and has minimum number of parameters (and delays) as in [8]. The advantages of this characterization are summarized in [11]. (A full journal article is scheduled to appear [13]). According to the 1-D results in [11], any $M \times M$ causal FIR lossless matrix $E(z)$ of McMillan degree K can be realized in the form

$$E(z) = V_K(z)V_{K-1}(z) \cdots V_1(z)H_0 \tag{2}$$

where H_0 is a constant $M \times M$ unitary matrix and $V_n(z)$ are $M \times M$ FIR lossless matrices of McMillan degree one, of the special form

$$V_n(z) = I - v_n v_n^\dagger + v_n v_n^\dagger z^{-1} \tag{3}$$

where v_n are unit-norm column vectors. The parameters of this characterization (i.e., H_0 and v_n) have been optimized in [11] to obtain analysis filters $H_k(z)$ with good attenuation characteristics. Note that any matrix of the form (3) where v_n has unit norm can be verified [11] to be lossless with determinant equal to z^{-1}.

For the 2-D case a causal FIR $E(z_1, z_2)$ is lossless if

$$\tilde{E}(z_1, z_2)E(z_1, z_2) = I \tag{4}$$

for all z_1, z_2. However a factorization analogous to (2) has not been established. Based on the fact that v_n has unit-norm, it is easy to prove that in the 2-D case $[I - v_n v_n^\dagger + v_n v_n^\dagger z_1^{-n_1} z_2^{-n_2}]$ is lossless for arbitrary integers $n_1, n_2 \geqslant 0$. And it is possible to cascade sections of this type to obtain 2-D non-separable FIR lossless systems of arbitrary degree (see [6] for further examples of 2-D FIR lossless systems). But none of these will result in a general structure that can realize arbitrary 2-D FIR lossless systems.

In this correspondence we shall restrict our attention to a particular sub-class of 2-D FIR lossless systems of the form

$$E(z_1, z_2) = \sum_{k_1=0}^{1} \sum_{k_2=0}^{K} e_{k_1,k_2} z_1^{-k_1} z_2^{-k_2}, \qquad K > 0. \tag{5}$$

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In other words, there is no restriction on the highest permissible power of z_2^{-1}, but the highest power of z_1^{-1} is unity. We shall then prove the following.

Lemma 1: Consider an $M \times M$ 2-D transfer matrix of the form (5). Assume the $e_{0,K}$ and $e_{1,K}$ are not *both zero*. Similarly assume $e_{0,0}$ and $e_{1,0}$ are not both zero. Then (5) is lossless if and only if it can be factorized into the form

$$E(z_1,z_2) = V_0(z_2) \cdots V_{J-1}(z_2) U(z_1) V_J(z_2) \cdots V_{K_2-1}(z_2) \quad (6)$$

for some integers J, K_2, where $V_n(z_2)$ are 1-D degree-one lossless systems of the form (3) in the variable z_2 and where $U(z_1)$ is a 1-D FIR lossless systems of the form (2) in the variable z_1.

Proof: If $D(z_1,z_2)$ denotes the determinant of $E(z_1,z_2)$, then we obtain $\tilde{D}(z_1,z_2)D(z_1,z_2)=1$ from (4). Combining this with the fact that $E(z_1,z_2)$ is causal FIR, we conclude that $D(z_1,z_2)$ has the form $D(z_1,z_2) = z_1^{-K_1} z_2^{-K_2}$, for integers $K_1, K_2 \geqslant 0$. Given such $E(z_1,z_2)$, we shall show how to find an $M \times 1$ unit-norm vector v and another causal $M \times M$ FIR lossless matrix $F(z_1,z_2)$ with determinant $z_1^{-K_1} z_2^{-(K_2-1)}$ such that either

$$E(z_1,z_2) = V(z_2) F(z_1,z_2) \quad (7)$$

or

$$E(z_1,z_2) = F(z_1,z_2) V(z_2) \quad (8)$$

where $V(z) = I - vv^{\dagger} + vv^{\dagger}z^{-1}$. This is called the *reduction step.* Essentially, we have "extracted" a degree-one lossless $V(z_2)$ from $E(z_1,z_2)$ to obtain a remainder lossless function $F(z_1,z_2)$. Since $V(z)$ is lossless, $V^{-1}(z) = \tilde{V}(z)$. Thus (7) is equivalent to

$$\left[I - vv^{\dagger} + vv^{\dagger}z_2\right] E(z_1,z_2) = F(z_1,z_2). \quad (9)$$

From (9) we see that $F(z_1,z_2)$ is paraunitary; moreover since the determinant of $V(z) = z^{-1}$, the determinant of $F(z_1,z_2)$ is $z_1^{-K_1} z_2^{-(K_2-1)}$. Same comments are true of $F(z_1,z_2)$ if it is obtained from (8). In order for $F(z_1,z_2)$ to be causal in (9) it is necessary and sufficient to satisfy

$$v^{\dagger} e_{0,0} = v^{\dagger} e_{1,0} = \mathbf{0}. \quad (10)$$

Similarly, if (8) has to hold for causal $F(z_1,z_2)$, it is necessary and sufficient to satisfy

$$e_{0,0} v = e_{1,0} v = \mathbf{0}. \quad (11)$$

We now show that there exists $v \neq \mathbf{0}$ such that at least one of the two conditions (10), (11) is satisfied. From the paraunitary property (4) we obtain the following three conditions by equating the coefficients of $z_1 z_2^K$, $z_1^{-1} z_2^K$, and z_2^K, respectively, to zero:

$$e_{1,K}^{\dagger} e_{0,0} = \mathbf{0} \quad (12a)$$

$$e^{\dagger}, K e_{1,0} = \mathbf{0} \quad (12b)$$

$$e_{0,K}^{\dagger} e_{0,0} + e_{1,K}^{\dagger} e_{1,0} = \mathbf{0}. \quad (12c)$$

Now the losslessness of $E(z_1,z_2)$, which implies (4), also implies $E(z_1,z_2)\tilde{E}(z_1,z_2) = I$, because $E(z_1,z_2)$ is a square matrix. From here we obtain the following three conditions by equating the coefficients of $z_1 z_2^K$, $z_1^{-1} z_2^K$ and z_2^K, respectively, to zero.

$$e_{0,0} e_{1,K}^{\dagger} = \mathbf{0} \quad (12d)$$

$$e_{1,0} e_{0,K}^{\dagger} = \mathbf{0} \quad (12e)$$

$$e_{0,0} e_{0,K}^{\dagger} + e_{1,0} e_{1,K}^{\dagger} = \mathbf{0}. \quad (12f)$$

By statement of Lemma 1, $e_{0,K}$ and $e_{1,K}$ cannot both be zero. We then have three cases.

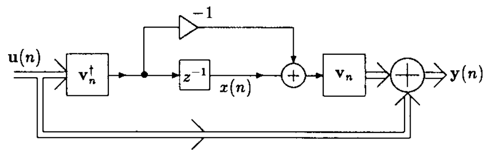

Fig. 1. An implementation of the degree-one lossless system $V_n(z)$.

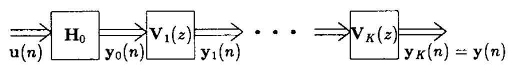

Fig. 2. The cascade-form realization of 1-D lossless $E(z)$, of degree K.

Case (i): Suppose $e_{1,K} = \mathbf{0}$. Then $e_{0,K} \neq \mathbf{0}$ by statement of lemma. From (12b) and (12c) we immediately see that (10) can be satisfied by taking v to be any nonzero column of $e_{0,K}$.

Case (ii): Suppose $e_{1,K} \neq \mathbf{0}$ but $e_{1,K}^{\dagger} e_{1,0} = \mathbf{0}$. From (12a) we also have $e_{1,K}^{\dagger} e_{0,0} = \mathbf{0}$. So by taking v to be any nonzero column of $e_{1,K}$ we can satisfy (10).

Case (iii): Suppose finally that $e_{1,K} \neq \mathbf{0}$ and $e_{1,K}^{\dagger} e_{1,0} \neq \mathbf{0}$. Postmultiplying (12f) with $e_{0,0}$ and using (12a) we get $e_{0,0} e_{0,K}^{\dagger} e_{0,0} = \mathbf{0}$. By using (12e) with this we see that (11) can be satisfied by taking v to be any column of $e_{0,K}^{\dagger} e_{0,0}$. There is guaranteed to exist a nonzero v because under this case $e_{0,K}^{\dagger} e_{0,0} \neq \mathbf{0}$ as seen from (12c).

Summarizing, we can always find $v \neq \mathbf{0}$ such that either (10) or (11) holds. We can always normalize this v so that it has unit norm and use it in (7) or (8) to obtain $F(z_1,z_2)$. Both $E(z_1,z_2)$ and $F(z_1,z_2)$ are causal $M \times M$ FIR lossless matrices with determinants equal to $z_1^{-K_1} z_2^{-K_2}$ and $z_1^{-K_1} z_2^{-(K_2-1)}$, respectively. If we repeat this operation K_2 times we obtain the factorization of the form

$$E(z_1,z_2) = V_0(z_2) \cdots V_{J-1}(z_2) W(z_1,z_2) V_J(z_2) \cdots V_{K_2-1}(z_2) \quad (13)$$

where $W(z_1,z_2)$ is causal $M \times M$ FIR lossless with determinant $z_1^{-K_1}$. We now claim that $W(z_1,z_2)$ is a function of z_1 only. To see this, note that if we perform the reduction step one more time on $W(z_1,z_2)$, then we obtain a remainder which is causal FIR lossless with determinant $z_1^{-K_1} z_2$. This is not possible because the remainder itself is a polynomial in negative powers of z_1 and z_2. Thus $W(z_1,z_2)$ is a function of z_1 only so that the form (6) has been established. This completes the proof of Lemma 1.

We now return to the question of state-space descriptions. We shall show that Roesser's state-space description [12] corresponding to (6) does have unitary system matrix, provided $U(z_1)$ is realized using a structure similar to (2). For this we first prove a 1-D result (which has *not* been observed in [11]):

Lemma 2: Let $E(z)$ be causal FIR $M \times M$ lossless implemented as in (2) with $V_n(z)$ as in (3). Then the system matrix R in (1) is unitary.

To prove this, first consider a system whose transfer matrix is (3). The system can be realized with a single dealy as in Fig. 1. The system matrix R for Fig. 1 can be easily verified to be unitary by using the fact that v has unit norm. Since the system matrix is defined as in (1), we have therefore established the equality

$$|x(n+1)|^2 + y^{\dagger}(n) y(n) = |x(n)|^2 + u^{\dagger}(n) u(n). \quad (14)$$

Now consider the cascade of Fig. 2 which represents the factorization (2). The kth section $V_k(z)$ satisfies a relation of the form

$$|x_k(n+1)|^2 + y_k^\dagger(n)y_k(n) = |x_k(n)|^2 + y_{k-1}^\dagger(n)y_{k-1}(n), \quad 1 \leqslant k \leqslant K \quad (15)$$

where $x_k(n)$ is the state-variable (output of z^{-1}) in the structure for $V_k(z)$. Moreover $u^\dagger(n)u(n) = y_0^\dagger(n)y_0(n)$ because H_0 is unitary. By adding the K equations in (15) we then see that the following is true:

$$x^\dagger(n+1)x(n+1) + y^\dagger(n)y(n) = x^\dagger(n)x(n) + u^\dagger(n)u(n) \quad (16)$$

for every possible initial state-vector $x(n)$ and for every possible current-input vector $u(n)$. Because of the relation

$$\begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix} = R \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}$$

this implies that R is indeed unitary.

Notice also that in the structure of Fig. 2 if we insert a constant unitary matrix between two adjacent $V_n(z)$'s, this does not change the unitary property of the system matrix. For the 2-D case, we shall consider the state-space description due to Roesser (as summarized in [12, eq. (2.1)]). The *horizontal state vector* is defined to be the vector of outputs of the elements z_1^{-1} in the structure, and will be denoted $x_1(n_1, n_2)$. The *vertical state vector* is defined to be the vector of outputs of the elements z_2^{-1} in the structure, and will be denoted $x_2(n_1, n_2)$. The state-space description is

$$\begin{bmatrix} x_1(n_1+1, n_2) \\ x_2(n_1, n_2+1) \end{bmatrix} = A \begin{bmatrix} x_1(n_1, n_2) \\ x_2(n_1, n_2) \end{bmatrix} + Bu(n_1, n_2) \quad (17)$$

$$y(n_1, n_2) = C \begin{bmatrix} x_1(n_1, n_2) \\ x_2(n_1, n_2) \end{bmatrix} + Du(n_1, n_2). \quad (18)$$

The system matrix is again defined as in (1). In order to prove the unitariness of the system matrix for the structure which realizes $E(z_1, z_2)$, note that if we set $z_1 = z_2 = z$ then (6) reduces to a 1-D FIR lossless matrix. The system matrix R remains unchanged by this substitution. By Lemma 2 this matrix is unitary (assuming that $U(z_1)$ is realized as a cascade similar to (2)), which proves that the system matrix R for the structure described by (6) is unitary.

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Integrator Neurons for Analog Neural Networks

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Symbols

N	Number of neurons.
M	Number of memory vectors.
$x_i(t)$	State of ith neuron at time t.
$u_i(t)$	Internal state of ith neuron at time t.
s_i^m	ith component of mth memory vector.
w_{ij}	Synaptic weight from jth neuron to ith neuron.
a_i	Amplification ratio of an amplifier of ith neuron in amplifier-neuron network (ANN).
A_i	Inverse of the time constant of an integrator of ith neuron in integrator-neuron network (INN).
f	Amplifier characteristic function for ANN.
g	Inverse function of f.
ϕ	Potential function defined for ANN.
Φ	Potential function defined for INN.
$l^m(t)$	Direction cosine of the state of the network ($x_i(t)$'s) to mth memorized vector.

I. Introduction

Analog amplifiers with saturation property can be used as neurons for analog neural networks. For auto-correlation type associative memory networks composed of these neurons, one can define a potential function such that it never increases as time proceeds [1]. Auto-correlation type associative memory network is a type of network used to solve optimization problems [2], [3]. We show below that integrators with saturation property can be used as neurons instead of amplifiers with saturation property, by defining a potential function, and that integrators are useful in wider parameter region than amplifiers if used for auto-correlation type associative memory networks. Hopfield [1] used pure amplifiers as output part of neurons which have sufficiently rapid response to internal states. Compared to that, integrators would be the other limit of the nature of neurons.

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